The binomial formula for nonsymmetric Macdonald polynomials

Siddhartha Sahi*
Department of Mathematics, Rutgers University, New Brunswick NJ 08903, USA

1. Introduction

The q-binomial theorem is essentially the expansion of $(x-1)(x-q)\cdots(x-q^{k-1})$ in terms of the monomials x^d. In a recent paper [O], A. Okounkov has proved a beautiful multivariate generalization of this in the context of symmetric Macdonald polynomials [M1]. These polynomials have nonsymmetric counterparts [M2] which are of substantial interest, and in this paper we establish nonsymmetric analogues of Okounkov's results.

An integral vector $v \in \mathbb{Z}^n$ is called "dominant" if $v_1 \geq \cdots \geq v_n$; and it is called a "composition" if $v_i \geq 0$, for all i. To avoid ambiguity we reserve the letters u, v for integral vectors, α, β, γ for compositions, and λ, μ for "partitions" (dominant compositions).

We write $|v|$ for $v_1 + \cdots + v_n$, and denote by w_v the (unique) shortest permutation in the symmetric group S_n such that $v^+ = w_v^{-1}(v)$ is dominant. Let $\mathbb{F}$ be field $\mathbb{Q}(q,t)$ where q, t are indeterminates. We write $\tau = (1, t^{-1}, \cdots, t^{-n+1})$, and define $\bar{v} = \bar{v}(q,t)$ in $\mathbb{F}^n$ by

$$\bar{v}_i = q^{v_i}(w_v\tau)_i.$$

Inhomogeneous analogues of nonsymmetric Macdonald polynomials were introduced in [K] and [S2]. They form an $\mathbb{F}$-basis for $\mathbb{F}[x] = \mathbb{F}[x_1, \cdots, x_n]$ and are defined as follows:

Definition: $G_\alpha \equiv G_\alpha(x; q, t)$ is the unique polynomial of degree $\leq |\alpha|$ in $\mathbb{F}[x]$ such that

1) the coefficient of $x^\alpha \equiv x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ in G_α is 1,
2) G_α vanishes at $x = \bar{\beta}$, for all compositions $\beta \neq \alpha$ such that $|\beta| \leq |\alpha|$.

As shown in Th. 3.9 of [K], the top homogeneous part of G_α is the nonsymmetric Macdonald polynomial E_α for the root system A_{n-1} ([M2] and [C]). Moreover by Th. 4.5 of [K] we have $G_\alpha(\bar{\beta}) = 0$ unless "$\alpha \subseteq \beta$". Here $\alpha \subseteq \beta$ means that if we write $w = w_\beta w_\alpha^{-1}$ then $\alpha_i < \beta_{w(i)}$ if $i < w(i)$ and $\alpha_i \leq \beta_{w(i)}$ if $i \geq w(i)$.

In this paper we obtain several new results about the polynomials G_α.

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Our first result is a formula for the special value $G_\alpha(a\bar{0}) = G_\alpha(a\tau) \in \mathbb{F}[a]$ where a is an indeterminate. This can be described in the following manner:

We identify α with the "diagram" consisting of points $(i,j) \in \mathbb{Z}^2$ with $1 \le i \le n$ and $1 \le j \le \alpha_i$. For $s = (i,j) \in \alpha$ we define the *arm*, *leg*, *coarm*, and *coleg* of s by

$$a(s) = \alpha_i - j,\ l(s) = \#\{k > i \mid j \le \alpha_k \le \alpha_i\} + \#\{k < i \mid j \le \alpha_k + 1 \le \alpha_i\},$$
$$a'(s) = j - 1,\ l'(s) = \#\{k > i \mid \alpha_k > \alpha_i\} + \#\{k < i \mid \alpha_k \ge \alpha_i\}.$$

1.1. Theorem. $G_\alpha(a\tau) = \prod_{s\in\alpha} \left(\dfrac{t^{1-n} - q^{a'(s)+1} t^{1-l'(s)}}{1 - q^{a(s)+1} t^{l(s)+1}} \right) \prod_{s\in\alpha} \left(a t^{l'(s)} - q^{a'(s)} \right).$

Let w_o be the longest element of S_n (which interchanges each i with $n-i+1$), and put $\widetilde{\beta} = \overline{-w_o\beta}$ and $\overline{\beta}^{-1} = \overline{\beta}(q^{-1}, t^{-1}) = (\overline{\beta}_1^{-1}, \cdots, \overline{\beta}_n^{-1})$. Then we have the following crucial "reciprocity" result:

1.2. Theorem. *There is a (unique) polynomial O_α of degree $\le |\alpha|$ in $\mathbb{Q}(q,t,a)[x]$ such that $O_\alpha(\overline{\beta}^{-1}) = G_\beta(a\widetilde{\alpha})/G_\beta(a\tau)$ for all β.*

We now introduce the following variants of G_α, which also form a basis for $\mathbb{F}[x]$:

Definition: $G'_\alpha = G'_\alpha(x; q, t)$ is the unique polynomial in $\mathbb{F}[x]$ such that

1) G'_α and G_α have the same top degree terms, i.e. E_α,
2) G'_α vanishes at $x = \widetilde{\beta}$ for all β with $|\beta| < |\alpha|$.

The existence of G'_α can be proved along the same lines as that of G_α ([K] Th. 2.3, [S2] Th. 4.3). One verifies that polynomials of degree $\le d$ are uniquely determined by their values at $x = \widetilde{\beta}$ for $|\beta| \le d$. Hence the lower degree terms of G'_α are determined by 2).

Definition: The "nonsymmetric (q,t)-binomial coefficients" are defined by

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix}_{q,t} = \frac{G_\beta(\overline{\alpha})}{G_\beta(\overline{\beta})} \equiv \frac{G_\beta(\overline{\alpha}(q,t); q, t)}{G_\beta(\overline{\beta}(q,t); q, t)}.$$

Our main result is the following relationship between G_α and G'_β:

1.3. Theorem. $\dfrac{G_\alpha(ax)}{G_\alpha(a\tau)} = \sum_{\beta\subseteq\alpha} a^{|\beta|} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}_{1/q,1/t} \dfrac{G'_\beta(x)}{G_\beta(a\tau)}.$

1.4. Corollary. $\dfrac{G_\alpha(x)}{G_\alpha(0)} = \sum_{\beta\subseteq\alpha} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}_{1/q,1/t} \dfrac{E_\beta(x)}{G_\beta(0)}.$ □

1.5. Corollary. $\dfrac{E_\alpha(x)}{E_\alpha(\tau)} = \displaystyle\sum_{\beta\subseteq\alpha} \begin{bmatrix}\alpha\\ \beta\end{bmatrix}_{1/q,1/t} \dfrac{G'_\beta(x)}{E_\beta(\tau)}.$ □

The corollaries follow from Theorem 1.3 by (1) replacing x by $a^{-1}x$ and letting $a \to 0$, and (2) by letting $a \to \infty$. For $n = 1$, Corollary 1.4 is essentially the q-binomial theorem.

If we put $t = q^r$ and let $q \to 1$ then $E_\alpha(x;r) \equiv \lim_{q\to 1} E_\alpha(x;q,q^r)$ is the nonsymmetric Jack polynomial [Op]. To discuss this limiting case, we define $\delta \equiv (0,-1,\cdots,-n+1)$, $\rho = r\delta$, and $\bar{\alpha}(r) = \alpha + w_\alpha\rho$.

Definition: $G_\alpha(x;r)$ is the unique polynomial of degree $\le |\alpha|$ in $\mathbb{Q}(r)[x]$ such that

1) the coefficient of $x^\alpha \equiv x_1^{\alpha_1}\cdots x_n^{\alpha_n}$ in $G_\alpha(x;r)$ is 1,
2) $G_\alpha(x;r)$ vanishes at $x = \bar{\beta}(r)$, for all compositions $\beta \neq \alpha$ such that $|\beta| \le |\alpha|$.

Definition: The "nonsymmetric r-binomial coefficients" are $\begin{bmatrix}\alpha\\ \beta\end{bmatrix}_r = \dfrac{G_\beta(\bar{\alpha}(r);r)}{G_\beta(\bar{\beta}(r);r)}.$

Definition: $G'_\alpha(x;r)$ is the unique polynomial in $\mathbb{Q}(r)[x]$ such that

1) $G'_\alpha(x;r)$ and $G_\alpha(x;r)$ have the same top degree terms,
2) $G'_\alpha(x;r)$ vanishes at $x = \widetilde{\beta}(r) \equiv \overline{-w_o\beta}(r)$ for all β with $|\beta| < |\alpha|$.

Theorems 1.1 — 1.3 have analogues in this setting.

If a is a scalar and x is a vector, write $a + x$ for $(a + x_1, \cdots, a + x_n)$.

1.6. Theorem. $G_\alpha(a+\rho;r) = \displaystyle\prod_{s\in\alpha}\left(\frac{a'(s)+1-rl'(s)+rn}{a(s)+1+rl(s)+r}\right)\prod_{s\in\alpha}(a - a'(s) + rl'(s)).$

1.7. Theorem. *There is a (unique) polynomial $O_\alpha(x;r)$ of degree $\le |\alpha|$ in $\mathbb{Q}(a,r)[x]$ such that we have $O_\alpha(\bar{\beta}(r);r) = G_\beta(a+\tilde{\alpha}(r);r)/G_\beta(a+\rho;r)$ for all β.*

1.8. Theorem. $\dfrac{G_\alpha(a+x;r)}{G_\alpha(a+\rho;r)} = \displaystyle\sum_{\beta\subseteq\alpha}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}_r \dfrac{G'_\beta(x;r)}{G_\beta(a+\rho;r)}.$

Since $\bar{\alpha}_i(r) = \lim_{q\to 1}(\bar{\alpha}_i(q,q^r) - 1)/(q-1)$, as in [K] Th. 6.2. we get $G_\alpha(x;r) \equiv \lim_{q\to 1} G_\alpha(1+(q-1)x;q,q^r)/(q-1)^{|\alpha|}$. It follows that the top terms of $G_\alpha(x;r)$ and $G'_\alpha(x;r)$ are $E_\alpha(x;r)$, and that $\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]_r = \lim_{q\to 1}\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]_{q,q^r} = \lim_{q\to 1}\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]_{1/q,1/q^r}$

So setting $x = ax$ and letting $a \to \infty$ in Theorem 1.8 we get

1.9. Corollary. $\dfrac{E_\alpha(1+x;r)}{E_\alpha(1;r)} = \displaystyle\sum_{\beta\subseteq\alpha}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}_r \dfrac{E_\beta(x;r)}{E_\beta(1;r)}.$ □

It seems to be difficult to deduce Theorems 1.6 — 1.8 directly from Theorems 1.1 — 1.3 by a limiting procedure. However the *proofs* of in the (q,t)-case *can* be modified to make them work in this setting.

We now describe some *new* phenomena in the limiting case. Write s_i for the transposition $(i\; i+1)$ which acts on $\mathbb{Q}(a)[x]$ by permuting x_i and x_{i+1}, and let

$$\sigma_i = s_i + \frac{r}{x_i - x_{i+1}}(1 - s_i).$$

Then, as observed in [K] Cor. 6.5, the map $\sigma : s_i \mapsto \sigma_i$ extends to a representation of S_n.

1.10. Theorem. $G'_\alpha(x;r) = (-1)^{|\alpha|}\sigma(w_o)w_o G_\alpha(-x-(n-1)r;r)$.

Using this, and writing $G^+_\alpha(x;r) := (-1)^{|\alpha|}G_\alpha(-x-(n-1)r;r)$, we get

1.11. Corollary. $\dfrac{\sigma(w_o)G_\alpha(a+x;r)}{G_\alpha(a+\rho;r)} = \displaystyle\sum_{\beta\subseteq\alpha}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}_r \dfrac{w_o G^+_\beta(x;r)}{G_\beta(a+\rho;r)}.$ □

As mentioned earlier, the symmetric analogues of Theorems 1.1 — 1.3 have been established in [O]. In the case of symmetric Jack polynomials, expansions in the form of Corollary 1.9 were first considered by [B] ($r=1/2$), and [Lc] ($r=1$), and in general by [Ls]. The analogues of Theorems 1.6, 1.7, and Corollary 1.9 have been obtained by [OO], but the analogues of Theorem 1.8, and 1.10 seem not to have been considered by them. Since these follow easily by our techniques, we shall formulate and prove them in Theorem 6.3 and Theorem 6.2 below.

While our proof follows the same general outline as Okounkov's argument, there are several differences. First, a decisive role is played by the affine Hecke algebra and Cherednik operators, and the Hecke recursions satisfied by the G_α actually yield a simplification of part of the argument. On the other hand, there are some subtleties in the nonsymmetric case, as exhibited by the definition of G'_α.

2. Preliminaries

We start by recalling certain basic properties of the $G_\alpha(x;q,t)$ (see [K] and [S2]).

The main result of [K] (Th. 3.6) is that the G_α satisfy the eigen-equations

$$\Xi_i G_\alpha = \bar{\alpha}_i^{-1} G_\alpha$$

for the "inhomogeneous Cherednik operators" defined by

$$\Xi_i = x_i^{-1} + x_i^{-1} H_i \dots H_{n-1}\Phi H_1 \dots H_{i-1}.$$

In turn, the operators Φ and H_i are defined by

$$\Phi f(x_1,\cdots,x_n) = (x_n - t^{-n+1}) f(x_n/q, x_1,\cdots,x_{n-1})$$
$$H_i = ts_i - (1-t)\frac{x_i}{x_i - x_{i+1}}(1-s_i).$$

The H_i's satisfy the braid relations and the identity $(H_i - t)(H_i + 1) = 0$, and generate a representation of the Iwahori-Hecke algebra $\mathcal{H}$ of S_n on $\mathbb{F}[x]$.

Next, write $v^{\#} = (v_n - 1, v_1, \cdots, v_{n-1})$; and let a be an indeterminate.

2.1. Lemma.

1) $\Phi f(a\bar{v}) = (a\bar{v}_n - t^{-n+1}) f(a\overline{v^{\#}})$
2) $H_i f(a\bar{v}) = \dfrac{(t-1)\bar{v}_i}{\bar{v}_i - \bar{v}_{i+1}} f(a\bar{v}) + \dfrac{\bar{v}_i - t\bar{v}_{i+1}}{\bar{v}_i - \bar{v}_{i+1}} f(a\overline{s_i v})$.

This is proved just as in [K] Lemmas 2.1, 3.1. The main point in 2) is that for $v \in \mathbb{Z}^n$, $s_i v = v \Rightarrow \bar{v}_i - t\bar{v}_{i+1} = 0$, and $s_i v \neq v \Rightarrow s_i\bar{v} = \overline{s_i v}$.

2.2. Lemma.

1) *If* $\alpha_n > 0$ *then* $G_\alpha = q^{\alpha_n - 1}\Phi G_{\alpha^{\#}}$.
2) *If* $\alpha_i > \alpha_{i+1}$ *then* $G_\alpha = (H_i + (1-t)d^{-1})G_{s_i\alpha}$ *where* $d = (1 - \bar{\alpha}_i/\bar{\alpha}_{i+1})$.

This is essentially in [K] and [S2] — here is a sketch of the argument: Evidently the right sides of 1) and 2) have degree $\leq |\alpha|$, and by using Lemma 2.1 one verifies the vanishing conditions. It remains only to check that the coefficient of x^α is 1. This obvious for 1), while for 2) one has to use the triangularity of Ξ_i (Lemma 3.10 of [K]).

In connection with Theorem 1.1, we define scalars $d_\alpha(q,t) = \prod_{s\in\alpha}(1 - q^{a(s)+1}t^{l(s)+1})$, $e_\alpha(q,t) = \prod_{s\in\alpha}(t^{1-n} - q^{a'(s)+1}t^{1-l'(s)})$, and $\phi_\alpha(a;q,t) = \prod_{s\in\alpha}\left(at^{l'(s)} - q^{a'(s)}\right)$.

2.3. Lemma.

1) *If* $\alpha_n > 0$ *then* $d_\alpha/d_{\alpha^{\#}} = 1 - t^n\bar{\alpha}_n$, $e_\alpha/e_{\alpha^{\#}} = t^{1-n} - t\bar{\alpha}_n$, $\phi_\alpha(0) = -q^{\alpha_n - 1}\phi_{\alpha^{\#}}(0)$.
2) *If* $\alpha_i > \alpha_{i+1}$ *then* $d_\alpha = \frac{1-\bar{\alpha}_i/\bar{\alpha}_{i+1}}{1-t\bar{\alpha}_i/\bar{\alpha}_{i+1}} d_{s_i\alpha}$.
3) $e_{w\alpha} = e_\alpha$ *and* $\phi_{w\alpha} = \phi_\alpha$ *for all* w *in* S_n.

The lemma can be proved in a manner very similar to Lemmas 4.1 and 4.2 in [S3]. To illustrate the argument, we sketch the proof of $e_\alpha/e_{\alpha^{\#}} = t^{1-n} - t\bar{\alpha}_n$, other proofs are similar: It follows from the definition of $\bar{\alpha}$ that $\bar{\alpha}_i = q^{\alpha_i}t^{-k_i}$ where $k_i = \#\{k < i \mid \alpha_k \geq \alpha_i\} + \#\{k > i \mid \alpha_k > \alpha_i\}$.

The diagram of α is obtained from $\alpha^{\#}$ by adding a point to the end of the first row, and moving this row to the last place. The new point $s = (n, \alpha_n) \in \alpha$ has $a'(s) = \alpha_n - 1$ and $l'(s) = \#\{k < n \mid \alpha_k > \alpha_n\} = k_n$, while coarms and colegs of other points are unchanged. Thus $e_\alpha / e_{\alpha^\#} = t^{1-n} - q^{a'(s)+1} t^{1-l'(s)} = t^{1-n} - q^{\alpha_n} t^{1-k_n} = t^{1-n} - t\bar{\alpha}_n$.

We shall also need limit versions of these results which are proved similarly:

First, by [K] Th. 6.6, we know that the $G_\alpha(x;r)$ satisfy the eigen-equations

$$\widetilde{\Xi}_i G_\alpha(x;r) = \bar{\alpha}(r) G_\alpha,$$

where the "limit" Cherednik operators are defined by

$$\widetilde{\Xi}_i := x_i - \sigma_i \ldots \sigma_{n-1} \widetilde{\Phi} \sigma_1 \ldots \sigma_{i-1}.$$

where $\sigma_i = s_i + r(x_i - x_{i+1})^{-1}(1 - s_i)$ is as in the previous section, and

$$\widetilde{\Phi} f(x) = (x_n + (n-1)r) f(x_n - 1, x_1 \ldots, x_{n-1}).$$

2.4. Lemma.

1) $\widetilde{\Phi} f(a + \bar{v}) = (a + \bar{v}_n + nr - r) f(a + \overline{v^\#})$

2) $\sigma_i f(a + \bar{v}) = \dfrac{r}{\bar{v}_i - \bar{v}_{i+1}} f(a + \bar{v}) + \dfrac{\bar{v}_i - \bar{v}_{i+1} - r}{\bar{v}_i - \bar{v}_{i+1}} f(a + \overline{s_i v}).$ $\square$

2.5. Lemma.

1) *If* $\alpha_n > 0$ *then* $G_\alpha = \widetilde{\Phi} G_{\alpha^\#}$.

2) *If* $\alpha_i > \alpha_{i+1}$ *then* $G_\alpha = (\sigma_i + r d^{-1}) G_{s_i \alpha}$ *where* $d = \bar{\alpha}_i - \bar{\alpha}_{i+1}$. $\square$

In connection with Theorem 1.6 we define scalars $d_\alpha(r) = \prod_{s \in \alpha} (a(s) + 1 + rl(s) + r)$, $e_\alpha(r) = \prod_{s \in \alpha} (a'(s) + 1 - rl'(s) + rn)$, and $\phi_\alpha(a;r) = \prod_{s \in \alpha} (a - a'(s) + rl'(s))$.

2.6. Lemma.

1) *If* $\alpha_n > 0$ *then* $d_\alpha(r)/d_{\alpha^\#}(r) = rn + \bar{\alpha}_n(r) = e_\alpha(r)/e_{\alpha^\#}(r)$.

2) *If* $\alpha_i > \alpha_{i+1}$ *then* $d_\alpha(r) = \frac{d}{d+r} d_{s_i \alpha}(r)$, *where* $d = \bar{\alpha}_i - \bar{\alpha}_{i+1}$.

3) $e_{w\alpha}(r) = e_\alpha(r)$ *and* $\phi_{w\alpha} = \phi_\alpha$ *for all* w *in* S_n. $\square$

We now briefly discuss the symmetric case.

Definition: $R_\lambda(x; q, t)$ is the unique symmetric polynomial of degree $\leq |\lambda|$ which vanishes at $x = \bar{\mu}$ for partitions $\mu \neq \lambda, |\mu| \leq |\lambda|$ and is normalized so that the coefficient of x^λ is 1.

Definition: $R_\lambda(x;r)$ is the unique symmetric polynomial of degree $\le |\lambda|$ which vanishes at $x=\bar\mu(r)$ for partitions $\mu\ne\lambda, |\mu|\le|\lambda|$ and is normalized so that the coefficient of x^λ is 1.

The existence and uniqueness of $R_\lambda(x;q,t)$ was proved in [K] and [S2], as was the fact that its top term is the Macdonald polynomial $P_\lambda(q,t)$. In the case of $R_\lambda(x;r)$ these results were established in [S1] and [KS].

As in [S2] Th. 4.6 and [K] Cor. 2.6, we have:

2.7. Lemma. *Let V_λ be the $\mathbb{F}$-span of $\{E_\alpha(x;q,t)\mid \alpha^+=\lambda\}$. Then V_λ is a module for the Hecke algebra $\mathcal{H}$, and $V_\lambda^{\mathcal{H}}=\mathbb{F}R_\lambda(x;q,t)$.*

2.8. Lemma. *Let $V_\lambda(r)$ be the $\mathbb{Q}(r)$-span of $\{E_\alpha(x;r)\mid \alpha^+=\lambda\}$. Then $V_\lambda(r)$ is a module for $\sigma(S_n)$, and $V_\lambda(r)^{\sigma(S_n)}=\mathbb{Q}(r)R_\lambda(x;r)$.*

Finally, For compatibility of notation between [K], [O] and [S2] we point out that

1) [K] uses P_λ for R_λ, $\overline{P}_\lambda$ for P_λ, $\overline{E}_\alpha$ for E_α, and E_α for G_α.
2) [O] uses $P^*_\lambda(x)$ for the "(shifted)" polynomial $R_\lambda(x\tau)\equiv R_\lambda(x_1,x_2t^{-1},\cdots,x_nt^{1-n})$ which vanishes at $(q_1^\mu,\cdots,q_n^\mu)$ and is symmetric in the variables x_it^{-i}.
3) [S2] uses $R_\lambda(x;q,t)$ to denote the polynomial $t^{-(n-1)|\lambda|}R_\lambda(xt^{n-1};q^{-1},t^{-1})$, which is symmetric and vanishes at the points $x=(q^{-\mu_1}t^{-n+1},\cdots,q^{-\mu_{n-1}}t^{-1},q^{-\mu_n})$. Its top term is $P_\lambda(x;q^{-1},t^{-1})$ which equals $P_\lambda(x;q,t)$ by [M1].

3. Evaluation

In this section we prove the evaluation formulas Theorem 1.1 and Theorem 1.6.

3.1. Lemma. *For all $w\in S_n$, we have $d_{w\alpha}(q,t)G_{w\alpha}(a\tau;q,t)=d_\alpha(q,t)G_\alpha(a\tau;q,t)$.*

Proof: It suffices to verify this for $w=s_i$, and we may also assume that $\alpha_i>\alpha_{i+1}$.

Since $\tau=\bar 0$, substituting $v=0$ in 2) of Lemma 2.1 we get $(H_if)(a\tau)=tf(a\tau)$ for all functions f. Combining this with 2) of Lemma 2.2 we get

$$G_\alpha(a\tau)=(t+(1-t)d^{-1})G_{s_i\alpha}(a\tau)=\frac{1-t\bar\alpha_i/\bar\alpha_{i+1}}{1-\bar\alpha_i/\bar\alpha_{i+1}}G_{s_i\alpha}(a\tau).$$

The result now follows from part 2) of Lemma 2.3. □

Theorem 1.1 states $d_\alpha G_\alpha(a\tau)=e_\alpha\phi_\alpha(a\tau)$, and we first establish this for $a=0$.

3.2. Lemma. $d_\alpha G_\alpha(0)=e_\alpha\phi_\alpha(0)=e_\alpha\prod_{s\in\alpha}(-q^{a'(s)})$.

Proof: The case $\alpha = 0$ is trivial, and we proceed by induction on $|\alpha|$ assuming $\alpha \neq 0$. By Lemma 3.1 and part 3) of Lemma 2.3 both sides are unchanged if permute α, so we may assume that $\alpha_n > 0$ and that $d_{\alpha^\#} G_{\alpha^\#}(0) = e_{\alpha^\#} \phi_{\alpha^\#}(0)$. Thus it suffices to prove

$$\frac{G_\alpha(0)}{G_{\alpha^\#}(0)} = \left(\frac{e_\alpha}{e_{\alpha^\#}}\right)\left(\frac{d_{\alpha^\#}}{d_\alpha}\right)\left(\frac{\phi_\alpha(0)}{\phi_{\alpha^\#}(0)}\right).$$

The left side can be computed by combining 1) of Lemmas 2.1 and 2.2, and the right side can be computed by 1) of Lemma 2.3. In each case we get $-q^{\alpha_n - 1} t^{1-n}$. $\square$

We now deduce Theorem 1.1 from the symmetric case [O].

Proof: (of Theorem 1.1) If λ is a partition then, by [O] (formula (1.9)), $R_\lambda(a\tau)$ is an $\mathbb{F}$-multiple of $\phi_\lambda(a)$.

Next, if α be a composition such that $\alpha^+ = \lambda$, then by Lemma 2.7 there are some coefficients $c_w \in \mathbb{F}$ such that $R_\lambda(x) = \sum_{w \in S_n} c_w d_{w\alpha} G_{w\alpha}(x)$. Evaluating at $x = a\tau$ and using Lemma 3.1 we get $R_\lambda(a\tau) = (\sum c_w) d_\alpha G_\alpha(a\tau)$. It follows that $d_\alpha(q,t) G_\alpha(a\tau)$ is an $\mathbb{F}$-multiple of $\phi_\lambda(a) = \phi_\alpha(a)$.

Setting $a = 0$ and using Lemma 3.2 we see that this multiple is $e_\alpha(q,t)$ and Theorem 1.1 follows. $\square$

Proof: (of Theorem 1.6) Arguing as in Lemma 3.1 we deduce that $d_{w\alpha}(r) G_{w\alpha}(a+\rho; r) = d_\alpha(r) G_\alpha(a+\rho; r)$. Next, by formula 2.3 of [OO], $R_\lambda(a+\rho; r)$ is a $\mathbb{Q}(r)$- multiple of $\phi_\lambda(a; r)$. Arguing as before, we conclude that $d_\alpha(r) G_\alpha(a+\rho; r)$ is a $\mathbb{Q}(r)$-multiple of $\phi_\alpha(a; r)$.

Letting $a \to \infty$ we see that the multiple is $d_\alpha(r) E_\alpha(1; r)$ which equals $e_\alpha(r)$ by Th. 1.3 of [S3]. The result follows. $\square$

4. Reciprocity

In this section we prove Theorem 1.2 and Theorem 1.7.

Proof: (of Theorem 1.2) Write $\mathbb{K} = \mathbb{F}(a) = \mathbb{Q}(q,t,a)$. For f in $\mathbb{K}[x]$, we have

$$\Xi_i f(a\bar{v}) = (a\bar{v}_i)^{-1} f(a\bar{v}) + (a\bar{v}_i)^{-1} H_i \dots H_{n-1} \Phi H_1 \dots H_{i-1} f(a\bar{v}).$$

Since $|v^\#| = |v| - 1$ and $|s_i v| = |v|$, it follows from Lemma 2.1 that the second term on the right is a combination of $f(a\bar{u})$ with $|u| = |v| - 1$, where the coefficients do not depend on f. Thus if p is a polynomial of degree d, and we write $p(\Xi) \equiv p(\Xi_1, \dots, \Xi_n)$, then $p(\Xi) f(a\tau) \equiv p(\Xi) f(a\bar{0}) = \sum_{|\beta| \le d} c_p(\beta) f(a\widetilde{\beta})$, with coefficients $c_p(\beta)$ independent of f.

Let $\mathcal{P}$ be the space of polynomials in $\mathbb{K}[x]$ of degree $\leq d$ and let $\mathcal{S}$ be the set of compositions β in $\mathbb{Z}_+^n$ with $|\beta| \leq d$. Then $p \mapsto c_p$ is a $\mathbb{K}$-linear map from $\mathcal{P}$ to $\mathbb{K}^{\mathcal{S}}$, and we claim that this map is bijective.

Since the spaces have the same dimension, it suffices to check injectivity. If $c_p = 0$ then $p(\Xi)f(a\tau) = 0$ for all f. In particular, setting $f = G_\beta$ we obtain $p(\overline{\beta}^{-1})G_\beta(a\tau) = 0$. By Theorem 1.1 $G_\beta(a\tau) \neq 0$, and it follows that p vanishes at the points $\overline{\beta}^{-1} = \overline{\beta}(q^{-1}, t^{-1})$ for all β, and hence $p = 0$, proving injectivity.

Now fix α with $|\alpha| = d$, and let O_α be the polynomial in $\mathcal{P}$ whose image under $p \mapsto c_p$ is the delta function at α in $\mathbb{K}^{\mathcal{S}}$. Then O_α has degree $\leq |\alpha|$, and satisfies $O_\alpha(\Xi)f(a\tau) = f(a\widetilde{\alpha})$ for all f. Setting $f = G_\beta$ we get $O_\alpha(\overline{\beta}^{-1})G_\beta(a\tau) = G_\beta(a\widetilde{\alpha})$. ☐

Proof: (Of Theorem 1.7) This is proved similarly by using the limit Cherednik operators $\widetilde{\Xi}_i$ and Lemma 2.4. ☐

5. The binomial formula

We now prove Theorem 1.3 and Theorem 1.8.

Proof: (of Theorem 1.3) Since the G'_β form a basis for $\mathbb{K}[x]$, there exist $b_{\beta\alpha} \in \mathbb{K}$ such that

$$(*) \qquad \frac{G_\alpha(ax)}{G_\alpha(a\tau)} = \sum_{\beta:|\beta|\leq|\alpha|} b_{\beta\alpha} G'_\beta(x).$$

Substituting $x = \widetilde{\gamma}$ and using Theorem 1.2 we get $O_\gamma(\overline{\alpha}^{-1}) = \sum_\beta b_{\beta\alpha} G'_\beta(\widetilde{\gamma})$.

Let G be the (infinite) matrix whose entries are $g_{\gamma\beta} = G'_\beta(\widetilde{\gamma})$. By Th. 4.2 in [S2] polynomials of degree $\leq d$ are determined by their values at the points $\{\widetilde{\gamma} : |\gamma| \leq d\}$, and it follows that G has an inverse H, and we get $b_{\beta\alpha} = \sum_\gamma h_{\beta\gamma} O_\gamma(\overline{\alpha}^{-1})$. Since $G'_\beta(\widetilde{\alpha}) = 0$ for $|\alpha| < |\beta|$ it follows that G and H are block triangular. Thus $h_{\beta\gamma} = 0$ for $|\gamma| > |\beta|$ and we deduce that $b_{\beta\alpha} = b_\beta(\overline{\alpha}^{-1})$ where $b_\beta := \sum_{\gamma:|\gamma|\leq|\beta|} h_{\beta\gamma} O_\gamma$ is a polynomial of degree $\leq |\beta|$.

The top degree term on the left side of $(*)$ is a multiple of E_α, and so by the definition of G'_α we obtain that $b_{\beta\alpha} = 0$ for $|\alpha| \leq |\beta|, \alpha \neq \beta$. Thus $b_\beta(\overline{\alpha}^{-1}) = 0$ for $|\alpha| \leq |\beta|, \alpha \neq \beta$, and since $\overline{\alpha}^{-1} = \overline{\alpha}(q^{-1}, t^{-1})$, it follows that $b_\beta(x)$ is a multiple of $G_\beta(x; q^{-1}, t^{-1})$. In other words, there are scalars c_β in $\mathbb{K}$ such that

$$\frac{G_\alpha(ax)}{G_\alpha(a\tau)} = \sum_\beta c_\beta \begin{bmatrix}\alpha \\ \beta\end{bmatrix}_{1/q,1/t} G'_\beta(x).$$

Comparing the top degree terms we get $c_\alpha = a^{|\alpha|}/G_\alpha(a\tau)$ and the result follows. ☐

Proof: (of Theorem 1.8) The proof proceeds similarly using Theorem 1.7. ☐

6. More on the Jack limit

We now prove Theorem 1.10 and the symmetric versions of Theorem 1.8 and Theorem 1.10. Since the (q,t)-case will not be considered in this section, we will often suppress r to simplify the notation, e.g. we will write $G_\alpha(x)$ for $G_\alpha(x;r)$, $\widetilde{\beta}$ for $\widetilde{\beta}(r)$ etc.

We start with a simple, but crucial, lemma.

6.1. Lemma. *1) $w_{-w_o\beta} = w_o w_\beta w_o$; 2) $-w_o\widetilde{\beta} = \overline{\beta} + (n-1)r$.*

Proof: For w in S_n, we have $(w_o w w_o)^{-1}(-w_o\beta) = (-w_o)(w^{-1}\beta)$, which is dominant if and only if $w^{-1}\beta$ is dominant. Since conjugation by w_o preserves length, part 1) follows.

Now $\widetilde{\beta} = \overline{-w_o\beta} = -w_o\beta + w_{-w_o\beta}\rho = -w_o\beta + w_o w_\beta w_o \rho$, by part 1). Also, since $w_o\rho = -(n-1)r - \rho$, we get $\widetilde{\beta} = -w_o(\beta + (n-1)r + w_\beta\rho) = -w_o(\overline{\beta} + (n-1)r)$. □

Proof: (of Theorem 1.10) For any polynomial f, wf and $\sigma(w)f$ have the same top terms. So, since $w_0^2 = 1$, the top term on the right of Theorem 1.10 is $(-1)^{|\alpha|} w_0^2 E_\alpha(-x) = E_\alpha(x)$. It remains only to show that the right side of Theorem 1.10 belongs to the space V consisting of polynomials which vanish at the points $x = \widetilde{\beta}, |\beta| < |\alpha|$.

Putting $a = 0$ and $v = \beta$ in 2) of Lemma 2.5, we deduce that V is σ-invariant and so it suffices to prove that $f \equiv w_o G_\alpha(-x-(n-1)r) \in V$. But, using Lemma 6.1 we get

$$f(\widetilde{\beta}) = w_o G_\alpha\left(-\widetilde{\beta} - (n-1)r\right) = G_\alpha\left(-w_o\widetilde{\beta} - (n-1)r\right) = G_\alpha(\overline{\beta}),$$

which vanishes for $|\beta| < |\alpha|$ by the definition of G_α. □

We now turn to the symmetric versions of Theorem 1.8 and Theorem 1.10. As in [OO] we define the "symmetric r-binomial coefficients" by

$$\binom{\lambda}{\mu}_r = \frac{R_\mu(\overline{\lambda}(r);r)}{R_\mu(\overline{\mu}(r);r)}.$$

The main result of [OO] is the generalized binomial formula

$$(**) \qquad \frac{P_\lambda(1+x;r)}{P_\lambda(1;r)} = \sum_{\mu\subseteq\lambda} \binom{\lambda}{\mu}_r \frac{P_\mu(x;r)}{P_\mu(1;r)}.$$

For the inhomogeneous analogue of this result, we define

Definition: $R'_\lambda(x;r)$ is the unique symmetric polynomial in $\mathbb{Q}(r)[x]$ such that

1) $R'_\lambda(x;r)$ and $R_\lambda(x;r)$ have the same top degree terms,
2) $R'_\lambda(x;r)$ vanishes at $x = \widetilde{\mu}(r) \equiv \overline{-w_o\mu}(r)$ for all μ with $|\mu| < |\lambda|$.

Then we have

6.2. Theorem. $R'_\lambda(x;r) = (-1)^{|\lambda|} R_\lambda(-x-(n-1)r;r)$.

Proof: The two sides have the same top degree terms, and it suffices to prove that the right side vanishes for $x=\widetilde{\mu}$ if $|\mu|<|\lambda|$. By symmetry, we may consider instead $x=w_o\widetilde{\mu}$. Substituting this and using Lemma 6.1, the right side becomes $(-1)^{|\lambda|}R_\lambda(\overline{\mu};r)$, which vanishes by definition of R_λ. $\square$

6.3. Theorem. $\dfrac{R_\lambda(a+x;r)}{R_\lambda(a+\rho;r)} = \sum_{\mu\subseteq\lambda} \binom{\lambda}{\mu}_r \dfrac{R'_\mu(x;r)}{R_\mu(a+\rho;r)}$.

We shall deduce Theorem 6.3 from Theorem 1.8 by symmetrization. Write $\mathcal{S}$ for the operator $\frac{1}{n!}\sum_{w\in S_n}\sigma(w)$ acting on $\mathbb{Q}(r)[x]$.

6.4. Lemma. *$\mathcal{S}$ maps polynomials to symmetric polynomials.*

Proof: For all i, we have $\sigma_i\mathcal{S}=\sum_{w\in S_n}\sigma(s_iw)=\mathcal{S}$. So if f is a polynomial in the image of $\mathcal{S}$, then $(1-\sigma_i)f=0$. Rewriting this we get $(1-\frac{r}{x_i-x_{i+1}})(1-s_i)f=0$. Hence $(1-s_i)f=0$ for all i, which implies that f is symmetric. $\square$

6.5. Lemma. *Let α be any composition with $\alpha^+=\lambda$, then*

$$1)\ \ \frac{\mathcal{S}G_\alpha(a+x)}{G_\alpha(a+\rho)}=\frac{R_\lambda(a+x)}{R_\lambda(a+\rho)};\quad 2)\ \ \frac{\mathcal{S}G'_\alpha(x)}{G_\alpha(a+\rho)}=\frac{R'_\lambda(x)}{R_\lambda(a+\rho)}.$$

Proof: If $|\beta|\le|\alpha|$ and $\beta^+\neq\alpha^+$, then Lemma 2.4 implies that, for all w in S_n, the polynomial $\sigma(w)G_\alpha(x)$ vanishes at $x=\overline{\beta}$. This means that $f=\mathcal{S}G_\alpha(a+x)$ vanishes at $\overline{\mu}-a$ for all partitions μ satisfying $|\mu|\le|\lambda|, \mu\neq\lambda$. Since f is symmetric and of the right degree, we conclude that f is a multiple of $R_\lambda(a+x)$. To determine the multiple we merely evaluate both sides of 1) at $x=\rho$, and use the fact that $\sigma(w)G_\alpha(a+\rho)=G_\alpha(a+\rho)$ which follows from 2) of Lemma 2.4. This proves 1).

For 2), the same argument proves that $\mathcal{S}G'_\alpha(x)$ vanishes at $\widetilde{\mu}$ for $|\mu|<|\lambda|$. To finish the proof, it suffices to prove that the *top* terms of the two sides are equal. But these are also the top terms of 1) and hence are equal. $\square$

Proof: (of Theorem 6.3) Fix α with $\alpha^+=\lambda$ and apply $\mathcal{S}$ to both sides of Theorem 1.8. By Lemma 6.5 we get

$$\frac{R_\lambda(a+x;r)}{R_\lambda(a+\rho;r)}=\sum_{\mu\subseteq\lambda}k_\mu\frac{R'_\mu(x;r)}{R_\mu(a+\rho;r)}\ \text{ with }\ k_\mu=\sum_{\beta^+=\mu}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}_r\in\mathbb{Q}(r).$$

To conclude we need to establish that $k_\mu = \binom{\lambda}{\mu}_r$, but this follows by putting $x = ax$ in the above, letting $a \to \infty$, and using $(**)$. □

6.6. Corollary. *For each α satisfying $\alpha^+ = \lambda$, we have* $\sum_{\beta^+=\mu} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}_r = \begin{pmatrix} \lambda \\ \mu \end{pmatrix}_r$. □

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